THE STRONG TREE PROPERTY AT SUCCESSORS OF SINGULAR CARDINALS (DRAFT)

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Abstract. We prove that successors of singular limits of strongly compact cardinals have the strong tree property. We also prove that $\aleph_{\omega+1}$ can consistently satisfy the strong tree property.

1. The Strong and the Super Tree Properties

We recall the definition of the tree property, for a regular cardinal κ.

Definition 1.1. *Let κ be a regular cardinal,*

(1) *a κ-tree is a tree of height κ with levels of size less than κ;*
(2) *we say that κ has the* tree property *if, and only if, every κ-tree has a cofinal branch (i.e. a branch of size κ).*

The strong and the super tree property concern special objects that generalize the notion of κ-tree, for a regular cardinal κ.

Definition 1.2. *Given $\kappa \geq \omega_2$ a regular cardinal and $\lambda \geq \kappa$, a (κ, λ)-*tree *is a set F satisfying the following properties:*

(1) *for every $f \in F$, $f : X \to 2$, for some $X \in [\lambda]^{<\kappa}$*
(2) *for all $f \in F$, if $X \subseteq \mathrm{dom}(f)$, then $f \restriction X \in F$;*
(3) *the set $\mathrm{Lev}_X(F) := \{f \in F;\ \mathrm{dom}(f) = X\}$ is non empty, for every $X \in [\lambda]^{<\kappa}$;*
(4) $|\mathrm{Lev}_X(F)| < \kappa$, *for all $X \in [\lambda]^{<\kappa}$.*

When there is no ambiguity, we will simply write Lev_X instead of $\mathrm{Lev}_X(F)$. The main difference between κ-trees and (κ, λ)-trees is the fact that, in the former, levels are indexed by ordinals, while in the latter, levels are indexed by *sets of ordinals*; ordinals are linearly ordered, sets of ordinals are not.

Definition 1.3. *Given $\kappa \geq \omega_2$ a regular cardinal, $\lambda \geq \kappa$, and a (κ, λ)-tree F,*

(1) *a* cofinal branch *for F is a function $b : \lambda \to 2$ such that $b \restriction X \in \mathrm{Lev}_X(F)$, for all $X \in [\lambda]^{<\kappa}$;*

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(2) *an* F-level sequence *is a function* $D : [\lambda]^{<\kappa} \to F$ *such that for every* $X \in [\lambda]^{<\kappa}$, $D(X) \in \mathrm{Lev}_X(F)$;
(3) *given an* F*-level sequence* D, *an* ineffable branch *for* D *is a cofinal branch* $b : \lambda \to 2$ *such that* $\{X \in [\lambda]^{<\kappa};\ b \restriction X = D(X)\}$ *is stationary.*

Definition 1.4. *Given* $\kappa \geq \omega_2$ *a regular cardinal and* $\lambda \geq \kappa$,

(1) (κ, λ)-TP *holds if every* (κ, λ)*-tree has a cofinal branch;*
(2) (κ, λ)-ITP *holds if for every* (κ, λ)*-tree* F *and for every* F*-level sequence* D, *there is an an ineffable branch for* D;
(3) *we say that* κ *satisfies the* strong tree property *if* (κ, μ)-TP *holds, for all* $\mu \geq \kappa$;
(4) *we say that* κ *satisfies the* super tree property *if* (κ, μ)-ITP *holds, for all* $\mu \geq \kappa$;

2. A Partition Property for Strongly Compact Cardinals

Let μ be a regular cardinal and $\lambda \geq \mu$ any ordinal. For every set $S \subseteq [\lambda]^{<\mu}$ cofinal we denote by $[[\ S\ ]]^2$ the set of all pairs $(X, Y) \in S \times S$ such that $X \subseteq Y$.

Definition 2.1. *Let* $\mu > \kappa$ *be two regular cardinals and let* $S \subseteq [\lambda]^{<\mu}$ *be a cofinal set and* $c : [[\ S\ ]]^2 \to \gamma$ *a function such that* $\gamma < \kappa$. *We say that a cofinal set* $H \subseteq S$ *is a* quasi homogenous *set of color* $i < \gamma$ *iff for every* $X, Y \in H$ *there is* $W \supseteq X, Y$ *in* H *such that* $c(X, W) = i = c(Y, W)$.

Theorem 2.2. *Let* κ *be a strongly compact, and let* $\nu > \kappa$ *be a cardinal. If* $\lambda \geq \nu^+$ *and* $S \subseteq [\lambda]^{<\nu^+}$ *is a stationary set, then every function* $c : [[\ S\ ]]^2 \to \gamma$, *with* $\gamma < \kappa$, *has a quasi homogenous set* H *which is also stationary.*

Proof. Let $\mathscr{U}$ be a $<\kappa$-complete ultrafilter containing every set $C \cap S$ with $C \subseteq [\lambda]^{<\nu^+}$ club – such ultrafilter exists since κ is strongly compact. Note that every set of $\mathscr{U}$ is stationary. First we show that for every $X \in S$, there is $i_X < \gamma$ and $H_X \subseteq S$ in $\mathscr{U}$ such that every Y in H_X contains X as a subset and $c(X, Y) = i_X$. Assume by contradiction that for every $i < \gamma$, the set $K_i := \{Y \in S;\ Y \supseteq X \text{ and } c(X, Y) \neq i\} \in \mathscr{U}$ then, from the $<\kappa$-completeness of $\mathscr{U}$, we have $\bigcap_{i<\gamma} C_i \in \mathscr{U}$, yet this is an empty set, a contradiction. From the fact that the ultrafilter is $<\kappa$-complete, we also get that the function $X \mapsto i_X$ is constant on a set $H \in \mathscr{U}$; let i be such that $i = i_X$, for every $X \in H$. Now, it is easy to see that H is quasi-homogenous of color i. Indeed, if $X, Y \in H$, then $H_X \cap H_Y \cap H$ belongs to $\mathscr{U}$ and it is, therefore, non empty. Let $Z \in H_X \cap H_Y \cap H$, then we have $c(X, Z) = i = c(Y, Z)$ as required. $\square$

Remark 2.3. *In the statement of Theorem 2.2, one can replace* γ *by any set of size* γ.

We can consider the previous partition property as an independent property for a regular cardinal.

Definition 2.4. *For a regular cardinal κ we say that κ has the* strong partition property *if for every cardinal $\nu > \kappa$, for every $\lambda \geq \nu^+$ and for every stationary set $S \subseteq [\lambda]^{<\nu^+}$, every function $c : [[\ S\ ]]^2 \to \gamma$, with $\gamma < \kappa$, has a quasi homogenous set H which is also stationary.*

3. The Strong Tree Property at Successors of Singular Cardinals

The following result generalizes a theorem by Magidor and Shelah [8].

Theorem 3.1. *Let ν be a singular limit of strongly compact cardinals, then ν^+ has the strong tree property.*

Proof. Assume without loss of generality that ν has countable cofinality, so let $\langle \kappa_n \rangle_{n<\omega}$ be an increasing sequence of strongly compact cardinals such that $\nu = \lim_{n<\omega} \kappa_n$. Let $\mu \geq \nu^+$ and let F be a (ν^+, μ)-tree. For every $X \in [\mu]^{<\nu^+}$, we assume that $\mathrm{Lev}_X(F) = \{f_i^X;\ i < |\mathrm{Lev}_X(F)|\}$.

Lemma 3.2. *(Spine Lemma) There exists $n < \omega$ and a stationary set $S \subseteq [\mu]^{<\nu^+}$, such that for all $X, Y \in S$, there are $\zeta, \eta < \kappa_n$ with $f_\zeta^X \upharpoonright (X \cap Y) = f_\eta^Y \upharpoonright (X \cap Y)$.*

Proof. Given a function $f \in Lev_X$, we write $\#f = i$ for $i < \nu$, when $f = f_i^X$. Define $c : [[\ [\mu]^{<\nu^+}]]^2 \to \omega$ by $c(X, Y) = \min\{i;\ \#(f_0^Y \upharpoonright X) < \kappa_i\}$. Since κ_0 is strongly compact, by Theorem 2.2, there is $n < \omega$ and a stationary quasi homogenous set $S \subseteq [\mu]^{<\nu^+}$ of color n. Then, for every $X, Y \in S$, there is $Z \supseteq X, Y$ in S such that $c(X, Z) = n = c(Y, Z)$. This means that $\#(f_0^Z \upharpoonright X)$, $\#(f_0^Z \upharpoonright Y) < \kappa_n$. So, let $\zeta, \eta < \kappa_n$ be such that $f_0^Z \upharpoonright X = f_\zeta^X$ and $f_0^Z \upharpoonright Y = f_\eta^Y$, then $f_\zeta^X \upharpoonright (X \cap Y) = f_0^Z \upharpoonright (X \cap Y) = f_\eta^Y \upharpoonright (X \cap Y)$, as required. □

Let n and S be like in the Spine Lemma, we prove the following fact.

Lemma 3.3. *There is a cofinal $S' \subseteq S$ and an ordinal $\zeta < \kappa_n$ such that for all $X, Y \in S'$, we have $f_\zeta^X \upharpoonright (X \cap Y) = f_\zeta^Y \upharpoonright (X \cap Y)$ (the set S' is even stationary).*

Proof. For every $(X, Y) \in [[\ S\ ]]^2$, let $\bar{c}(X, Y)$ be the minimum couple $(\zeta, \eta) \in \kappa_n \times \kappa_n$, in the lexicografical order, such that $f_\eta^Y \upharpoonright X = f_\zeta^X$; the function is well defined by definition of n and S. Since κ_{n+1} is strongly compact, we can apply Theorem 2.2 to the function $\bar{c} : [[\ S\ ]]^2 \to \kappa_n \times \kappa_n$. Hence, there exists a quasi homogenous set S' of color $(\zeta, \eta) \in \kappa_n \times \kappa_n$. It follows that for every $X, Y \in S'$, there is $Z \supseteq X, Y$ in S' such that $\bar{c}(X, Z) = (\zeta, \eta) = \bar{c}(Y, Z)$. This means that $f_\eta^Z \upharpoonright X = f_\zeta^X$ and $f_\eta^Z \upharpoonright Y = f_\zeta^Y$, hence $f_\zeta^X \upharpoonright (X \cap Y) = f_\eta^Z \upharpoonright (X \cap Y) = f_\zeta^Y \upharpoonright (X \cap Y)$. □

Set $b = \bigcup_{X \in S'} f^X_\zeta$, by the previous lemma b is a function; b is a cofinal branch for F. $\square$

Remark 3.4. *Our proof of Theorem 3.1 makes it clear that even if ν is just a singular limit of regualr cardinals that satisfy the strong partition property, then ν^+ has the strong tree property.*

4. Systems

Definition 4.1. *Given $\lambda \geq \nu^+$, $D \subseteq [\lambda]^{<\nu^+}$ cofinal and $\mathscr{S} := \{S_i\}_{i \in I}$ a family of binary relations over $D \times \nu$, we say that $\mathscr{S}$ is a* system *if the following hold:*

1. *if $(X, \zeta)\ S_i\ (Y, \eta)$ and $(X, \zeta) \neq (Y, \eta)$, then $X \subsetneq Y$;*
2. *for every $X \subseteq Y$, if both $(X, \zeta)\ S_i\ (Z, \theta)$ and $(Y, \eta)\ S_i\ (Z, \theta)$, then $(X, \zeta)\ S_i\ (Y, \eta)$;*
3. *for every $X, Y \in D$, there is $Z \supseteq X, Y$ and $\zeta_X, \zeta_Y, \eta \in \nu$ such that for some $i \in I$ we have $(X, \zeta_X)\ S_i\ (Z, \eta)$ and $(Y, \zeta_Y)\ S_i\ (Z, \eta)$ (in particular, if $X \subseteq Y$, then $(X, \zeta)\ S_i\ (Y, \zeta_Y)$).*

In the previous definition, the elements of $D \times \nu$ are called *nodes*. Given two nodes u and v, we say that they are *S_i-incompatible,* for some $i \in I$, if there is no $w \in D \times \nu$ such that $u\ S_i\ w$ and $v\ S_i\ w$. Sometimes we will say that a node u belongs to the X'th level if the first coordinate of u is X (i.e. $u = (X, \zeta)$, for some $\zeta \in \nu$).

Definition 4.2. *Let $\{S_i\}_{i \in I}$ be a system on $D \times \nu$ and let $i \in I$, a partial function $b : D \to \nu$ is an S_i-*branch *if it satisfies the following conditions. For every $X \in \mathrm{dom}(b)$ and for every $Y \in D$ with $Y \subseteq X$, we have:*

1. *$Y \in \mathrm{dom}(b)$ iff there exists $\zeta < \nu$ such that $(Y, \zeta)\ S_i\ (X, b(X))$,*
2. *$b(Y)$ is the unique ζ witnessing this.*

In the situation of the previous definition, we say that a branch b is *cofinal* if $X \in \mathrm{dom}(X)$ for cofinally many X's.

Definition 4.3. *Let $\{S_i\}_{i \in I}$ be a system on $D \times \nu$, a* system of branches *is a family $\{b_j\}_{j \in J}$ such that*

1. *every b_j is an S_i-branch for some $i \in I$;*
2. *for every $X \in D$, there is $j \in J$ such that $X \in \mathrm{dom}(b_j)$.*

The following two results (Lemma 4.4 and Theorem 4.5) generalize a theorem by Sinapova (see [10] Preservation Lemma). The proof of this theorem is very similar to the proof of Sinapova's Preservation Lemma, we have just to deal with *sets of ordinals* instead of ordinals.

Lemma 4.4. *Let ν be a singular cardinal of countable cofinality, $\lambda \geq \nu$, and let $\{R_i\}_{i \in I}$ be a system on $D \times \tau$ with $D \subseteq [\lambda]^{<\nu^+}$ cofinal and $\max(|D|, \tau) < \nu$.*

Suppose that $\mathbb{P}$ is a κ-closed forcing, for a regular $\kappa > \max(|D|, \tau)^+$, and for some $p \in \mathbb{P}$, $\dot{b} \in V^{\mathbb{P}}$ and $i \in I$, we have

$$p \Vdash \dot{b} \text{ is a cofinal } R\text{-branch},$$

where $R = R_i$. If V has no cofinal branches for the system, then for all $\eta < \kappa$, we can find a sequence $\langle v_\zeta;\ \zeta < \eta\rangle$ of pairwise R-incompatible elements of $D \times \tau$ such that for every $\zeta < \eta$, there exists $q \leq p$ forcing $v_\zeta \in \dot{b}$.

Proof. Let $E := \{u \in D \times \tau;\ \exists q \leq p(q \Vdash u \in \dot{b})\}$. First remark that, since p forces that $\dot{b}$ is cofinal, the set $\{X \in D;\ \exists \zeta \in \tau\ (X, \zeta) \in E\}$ is cofinal. Since V has no cofinal branches for the system, we can find, for all $v \in E$ two R-incompatible nodes $w_1, w_2 \in E$ such that $v\ R\ w_1$, $v\ R\ w_2$.

We inductively define for all $\zeta < \eta$ two nodes $u_\zeta, v_\zeta \in E$ and a condition $p_\zeta \leq p$ such that:

(1) $p_\zeta \Vdash u_\zeta \in \dot{b}$;
(2) u_ζ and v_ζ are pairwise R-incompatible;
(3) for all $\varepsilon < \zeta$, $u_\varepsilon\ R\ u_\zeta$ and $u_\varepsilon\ R\ v_\zeta$;
(4) the sequence $\langle p_\varepsilon;\ \varepsilon \leq \zeta\rangle$ is decreasing;

Let u be any node in E. From the remark above, there are $u_0, v_0 \in E$ which are R-incompatible and both $u\ R\ u_0$ and $u\ R\ v_0$. By definition of E, there is a condition $p_0 \leq p$ such that $p_0 \Vdash u_0 \in \dot{b}$.

Let $\zeta > 0$ and assume that u_ε, v_ε, p_ε are defined for every $\varepsilon < \zeta$. Let q be stronger than every condition in $\{p_\varepsilon;\ \varepsilon < \zeta\}$. By the inductive hypothesis (claim 3.), the nodes $\langle u_\varepsilon;\ \varepsilon < \zeta\rangle$ form an R-chain, so we can find a node h and a condition $q^* \leq q$ such that $u_\varepsilon\ R\ h$, for all $\varepsilon < \zeta$, and $q^* \Vdash h \in \dot{b}$. Since p force that $\dot{b}$ is cofinal and there is no cofinal branch in V for the system, we can find two R-incompatible nodes $u_\zeta, v_\zeta \in E$ and a condition $p_\zeta \leq q^*$ such that $h\ R\ u_\zeta$, $h\ R\ v_\zeta$ and $p_\zeta \Vdash u_\zeta \in \dot{b}$. That completes the construction.

The sequence $\langle v_\zeta;\ \zeta < \eta\rangle$ is as required: for if $\zeta' < \zeta < \eta$, then by definition $u_{\zeta'}$ and $v_{\zeta'}$ are R-incompatible, and $u_{\zeta'}\ R\ v_\zeta$, hence $v_{\zeta'}$ and v_ζ are R-incompatible. □

Theorem 4.5. *(Preservation Theorem) Let ν be a singular cardinal of countable cofinality, $\lambda \geq \nu$, and let $\{R_i\}_{i\in I}$ be a system on $D \times \tau$ with $D \subseteq [\lambda]^{<\nu^+}$ cofinal and $\max(|D|, \tau) < \nu$. Suppose that $\mathbb{P}$ is a κ-closed forcing, for a regular $\kappa > \max(|D|, \tau)^+$, and assume that $\mathbb{P}$ forces a system of branches $\{\dot{b}_j\}_{j\in J}$ through $\{R_i\}_{i\in I}$ with $|J|^+ < \kappa$ and such that for some $j \in J$, the branch b_j is cofinal. Then, there exists in V a cofinal R_i-branch, for some $i \in I$.*

Proof. Suppose for contradiction that V has no cofinal branches for the system. We start working in $V[\mathbb{R}]$. Let $B := \{j \in J;\ b_j \text{ is not cofinal}\}$. By the closure of $\mathbb{P}$ (we have $|J|^+ < \kappa$), we can find a condition p deciding, for every $j \in J$ whether or not b_j is cofinal, hence $B \in V$. For every $j \in B$, fix $X_j \in [\lambda]^{<\nu^+}$ such that domb_j has empty intersection with every $Y \supseteq X_j$. Since B has size less than ν, the set $X^* \bigcup_{j \in B} X_j$ is in $[\lambda]^{<\nu^+}$. Let $C^* := \{Z \in D;\ X^* \subseteq Z\}$. Define $A := \{j \in J;\ p \Vdash \dot{b}_j \text{ is cofinal}\}$, then $A \in V$ and by hypothesis of the theorem A is non empty. Moreover, by strengthening p if necessary, we have $p \Vdash \forall X \in C^* \exists a \in A(X \in \mathrm{dom}(\dot{b}_j))$

(use condition (2) and the definition of C^*). Let η be a regular cardinal with $\max(|D|, \tau) < \eta < \kappa$.

Claim 4.6. *Let $\lhd$ be a well ordering of A. For every $a \in A$, we can define $\langle q^a_\gamma;\ \gamma < \eta\rangle$ and $\langle u^a_\gamma;\ \gamma < \eta\rangle$ such that*

(1) *for all $\gamma < \eta$, $q^a_\gamma \leq p$ and $q^a_\gamma \Vdash u^a_\gamma \in \dot{b}_a$,*
(2) *the nodes $\langle u^a_\gamma;\ \gamma < \eta\rangle$ are pairwise R-incompatible,*

in such a way that for all $\gamma < \eta$, $\langle q^a_\gamma;\ a \in A\rangle$ is $\lhd$-decreasing.

Proof. We proceed by induction on $a \in A$. Assume that the sequences have been defined up to $a \in A$. For every $\gamma < \eta$, let r_γ be stronger than every condition in the set $\{q^c_\gamma;\ c \lhd a\}$, and let $E_\gamma := \{u \in D \times \tau;\ \exists q \leq r_\gamma(q \Vdash u \in \dot{b}_a)\}$. For all $\gamma < \eta$, let $\langle v^\gamma_\zeta;\ \zeta < \eta\rangle$ be as in the conclusion of Lemma 4.4 applied to r_γ and $\dot{b}_a$, and let $X_\gamma \in [\chi]^{<\mu^+}$ be such that the level of each v^γ_ζ is below X_γ. Let $X^* \supsetneq \bigcup_{\gamma<\eta} X_\gamma$ in D. We want to define the sequence $\langle u^a_\gamma;\ \gamma < \eta\rangle$ with each $u^a_\gamma \in E_\gamma$ belonging to a level above X^*. We proceed by induction: suppose we have defined $\langle u^a_\gamma;\ \gamma < \delta\rangle$ for some $\delta < \eta$. For every $\gamma < \delta$, there is at most one $\zeta < \eta$ such that $v^\delta_\zeta\ R\ u^a_\gamma$ (because the v^δ_ζ's are pairwise R-incompatible). For all $\gamma < \delta$, let ζ_γ be that unique index if it exists and let ζ_γ be 0 otherwise. Choose $\zeta \in \eta \setminus \{\zeta^\delta_\gamma;\ \gamma < \delta\}$. Then, for all $\gamma < \delta$, the nodes v^δ_ζ and u_γ are R-incompatible. Let $u^a_\delta \in E_\delta$ be such that $v^\delta_\zeta\ R\ u^a_\delta$. Then, for all $\gamma < \delta$, the nodes u^a_γ and u^a_δ are R-incompatible. Since for every $\gamma < \eta$, we have $u^a_\gamma \in E_\gamma$, we can find a condition $q^a_\gamma \leq r_\gamma$ such that $q^a_\gamma \Vdash u^a_\gamma \in \dot{b}_a$. That completes the construction. □

We return to the proof of the theorem. For every $\gamma < \eta$, let p_γ be stronger than all the conditions $\langle q^a_\gamma;\ a \in A\rangle$, and let $Y_\gamma \in D$ be such that the nodes in $\{u^a_\gamma;\ a \in A\}$ belong to levels below Y_γ. Fix $Y^* \in C^*$ such that $Y^* \supseteq \bigcup_\gamma Y_\gamma$. For all $\gamma < \eta$, let $p^*_\gamma \leq p_\gamma$, let w_γ of level Y^* and $a_\gamma \in A$ such that $p^*_\gamma \Vdash w_\gamma \in \dot{b}_{a_\gamma}$. Since A has size less than η, there is w^* on level Y^* and $a^* \in A$ such that $w_\gamma = w^*$,

$a_\gamma = a^*$, for almost all $\gamma < \eta$. Let $b^* := \dot{b}_{a^*}$. Given two distinct $\gamma, \delta < \eta$ large enough, if $u := u_\gamma^{a^*}$ and $v := u_\delta^{a^*}$, then the following hold:

(1) $p^*_\gamma \Vdash u \in b^*$, $p^*_\gamma \Vdash w^* \in b^*$;
(2) $p^*_\delta \Vdash v \in b^*$, $p^*_\delta \Vdash w^* \in b^*$;
(3) both the level of u and the level of v are subsets of Y^*;
(4) u and v are R_{a^*}-incompatible,

that leads to a contradiction. $\square$

5. The Strong Tree Property at $\aleph_{\omega+1}$

Theorem 5.1. *Let $\langle \kappa_n \rangle_n < \omega$ be an increasing sequence of indestructibly supercompact cardinals. There is a strong limit cardinal $\mu < \kappa_0$ of cofinality ω such that by forcing over V with the poset*

$$\mathrm{Coll}(\omega, \mu) \times \mathrm{Coll}(\mu^+, < \kappa_0) \times \Pi_{n<\omega} \mathrm{Coll}(\kappa_n, < \kappa_{n+1}),$$

one gets a model where the strong tree property holds at $\aleph_{\omega+1}$.

Proof. Let κ denote κ_0, for every $\mu < \kappa$ we let

(1) $\mathbb{R}(\mu) := \mathrm{Coll}(\omega, \mu) \times \mathrm{Coll}(\mu^+, < \kappa_0) \times \Pi_{n<\omega} \mathrm{Coll}(\kappa_n, < \kappa_{n+1})$,
(2) $\mathbb{L}(\mu) := \mathrm{Coll}(\omega, \mu) \times \mathrm{Coll}(\mu^+, < \kappa_0)$,
(3) $\mathbb{C} := \Pi_{n<\omega} \mathrm{Coll}(\kappa_n, < \kappa_{n+1})$.

Assume that $\nu = \sup_n \kappa_n$, then the forcing $\mathbb{R}(\mu)$ produces a model where $\aleph_{\omega+1} = \nu^+$. We fix $H := \Pi_{n<\omega} H_n \subseteq \mathbb{C}$ generic over V. We work in $W := V[H]$. Assume for contradiction that in every extension of W by $\mathbb{L}(\mu)$ with $\mu < \kappa$ strong limit of cofinality ω, the strong tree property fails at ν^+. For every such μ, let λ_μ and $\dot{F}(\mu) \in W^{\mathbb{L}(\mu)}$ be a name for a (ν^+, λ_μ)-tree with no cofinal branches. Let $\lambda = \sup_{n<\omega} \lambda_\mu$, without loss of generality we can assume that $\lambda_\mu = \lambda$ for every μ, since a (ν^+, λ_μ)-tree with no cofinal branches can be extended to a (ν^+, λ)-tree with no cofinal branches. Given $X, Y \in [\lambda]^{<\nu^+}$ and $\zeta, \eta < \nu$, we will write $\Vdash_{\mathbb{L}(\mu)} (X, \zeta) <_{\dot{F}_\mu} (Y, \eta)$ when

$$\Vdash_{\mathbb{L}(\mu)} \text{ the } \eta\text{'th function on level } Y \text{ extends the } \zeta\text{'th function on level } X.$$

Consider the following set

$$I := \{(a, b, \mu);\ \mu < \kappa \text{ is strong limit of cof } \omega \text{ and } (a, b) \in \mathbb{L}(\mu)\}.$$

We define a system $\mathscr{S} = \{S_i\}_{i \in I}$ on $[\lambda]^{<\nu^+} \times \nu$ as follows. Given $i = (a, b, \mu) \in I$, for every $X, Y \in [\lambda]^{<\nu^+}$ and for every $\zeta, \eta < \nu$, we let

$$(X, \zeta)\ S_i\ (Y, \eta) \iff (a, b) \Vdash (X, \zeta) <_{\dot{F}_\mu} (Y, \eta).$$

Lemma 5.2. *There is, in W, an integer $n < \omega$ and a cofinal set $D \subseteq [\lambda]^{<\nu^+}$ such that $\{\ S_i \restriction D \times \kappa_n\}_{i \in I}$ is a system.*

Proof. Using the supercompactness of κ fix $j : W \to W^*$ an elementary embedding with critical point κ such that for σ large enough $j(\kappa) > \sigma$ and W^* is closed by sequences of length σ. In particular, $a^* := j[\lambda] \in W^* \cap [j(\lambda)]^{<j(\nu^+)}$.

Let $G_0^* \times G_1^*$ be generic for $\mathrm{Coll}(\omega, \nu)^{W^*} \times \mathrm{Coll}(\nu^+, < j(\kappa))^{W^*}$ over W, hence also over W^*. We let F^* be the interpretation of $j(\dot{F})(\nu)$ in $W[G_0^* \times G_1^*]$, where $\dot{F}$ is the map $\mu \mapsto \dot{F}(\mu)$. We denote by $\ll \lambda \gg^{<\nu^+}$ the set of all the strictly increasing sequences $X : \alpha \to \lambda$ such that $\alpha < \nu^+$. For every $X \in \ll \lambda \gg^{<\nu^+}$, the image of X, that we denote $Im(X)$, is a subset of $[\lambda]^{<\nu^+}$; by abuse of notation, we will sometime write just X instead of $Im(X)$. Now, we define a sequence $\langle (p_X, q_X, \zeta_X, n_X);\ X \in \ll \lambda \gg^{<\nu^+} \rangle$ such that

1. $(p_X, q_X) \in \mathrm{Coll}(\omega, \nu) \times \mathrm{Coll}(\nu^+, < j(\kappa))$, $n_X < \omega$, and $\zeta_X < j(\kappa_{n_X})$;
2. $(p_X, q_X) \Vdash (j[X], \zeta_X) <_{F^*} (a^*, 0)$;
3. for every $X \sqsubseteq Y$ in $\ll \lambda \gg^{<\nu^+}$, we have $q_Y \leq q_X$.

The sequence is inductively defined as follows. Let $X : \alpha \to \lambda$ be a strictly increasing sequence, assume by inductive hypothesis that $\langle (p_X, q_X, \zeta_X, n_X);\ X \in \ll \lambda \gg^{<\alpha} \rangle$ is defined. By condition (3), the sequence $\langle q_{X \restriction \beta};\ \beta < \alpha \rangle$ is decreasing and $\mathrm{Coll}(\nu^+, < j(\kappa))$ is ν^+-closed, so there exists a lower bound q_X. By strengthening q_X if necessary, we can find p_X, ζ_X, n_X satisfying conditions (1) and (2) for the sequence X. That completes the definition.

The poset $\mathrm{Coll}(\omega, \nu)$ has size less than $\lambda^{<\nu^+}$, hence there is a condition p and a cofinal set D such that for every $X \in D$, we have $p = p_{\bar{X}}$, where $\bar{X} \in \ll \lambda \gg^{<\nu^+}$ is the unique increasing sequence whose image is X. By shrinking D, we can also assume that there exists $n < \omega$ such that $n = n_{\bar{X}}$, for every $X \in D$.

Claim 5.3. *$\{\ S_i \restriction D \times \kappa_n\}_{i \in I}$ is a system.*

Proof. We just have to prove that it satisfies condition (3) of Definition 4.1. Fix $X, Y \in D$, then we have $(p_X, q_X) \Vdash (j[X], \zeta_X) <_{F^*} (a^*, 0)$ and $(p_Y, q_Y) \Vdash (j[Y], \zeta_Y) <_{F^*} (a^*, 0)$. If $Z \supseteq X, Y$ is any set in D, then q_Z is stronger than both q_X and q_Y. Therefore, condition (p, q_Z) forces that:

(i) $(j[X], \zeta_X) <_{F^*} (a^*, 0)$;
(ii) $(j[Z], \zeta_Z) <_{F^*} (a^*, 0)$;
(iii) $(j[Z], \zeta_Z) <_{F^*} (a^*, 0)$;
(iv) $(j[Y], \zeta_Y) <_{F^*} (a^*, 0)$.

From (i) and (ii) follows that $(p, q_Z) \Vdash (j[X], \zeta_X) <_{F^*} (j[Z], \zeta_Z)$; from (iii) and (iv) follows that $(p, q_Z) \Vdash (j[Y], \zeta_Y) <_{F^*} (j[Z], \zeta_Z)$. Then, by elementarity, there exists $\mu < \kappa$ and $(\bar{p}, \bar{q}) \in \mathbb{L}(\mu)$ and $\bar{\zeta}_X, \bar{\zeta}_Y, \bar{\zeta}_Z < \kappa_n$ such that $(\bar{p}, \bar{q}) \Vdash (X, \bar{\zeta}_X) <_{\dot{F}_\mu} (Z, \bar{\zeta}_Z)$ and $(Y, \bar{\zeta}_Y) <_{\dot{F}_\mu} (Z, \bar{\zeta}_Z)$. If we let $i = (\bar{p}, \bar{q}, \mu)$, then we just proved $(X, \bar{\zeta}_X)\ S_i\ (Z, \bar{\zeta}_Z)$ and $(Y, \zeta_Y)\ S_i\ (Z, \bar{\zeta}_Z)$. $\square$

That completes the proof of the lemma. □

To simplify the notation, we define $R_i := S_i \restriction D \times \kappa_n$, for every $i \in I$. Let $m = n+2$. By indestructibility of κ_m, forcing over $W = V[H]$ with $\mathrm{Coll}(\kappa_m, \gamma)^V$ for γ sufficiently large, adds an elementary embedding $\pi : V[H] \to M[H^*]$ with critical point κ_m and such that $\pi(\kappa_m)$ is large enough. So $H^* \subseteq \pi(\mathbb{C}) = \mathbb{C} * \mathbb{C}_{tail}$ where $\mathbb{C}_{tail}$ is κ_m-closed.

Lemma 5.4. *There is in $V[H^*]$ a system of branches $\{b_j\}_{j\in J}$ for the system $\{R_i\}_{i\in I}$ with $J = I \times \kappa_n$, such that for some $j \in J$, the branch b_j is cofinal.*

Proof. First note that since $\kappa_n, |I| < cr(\pi)$, we may assume that $\pi(I) = I$ and $\pi(\{R_i\}_{i\in I}) = \{\pi(R_i)\}_{i\in I}$. This is a system on $\pi(D) \times \kappa_n$. Let a^* be a set in $\pi(D)$ such that $\pi[\lambda] \subseteq a^*$. For every $(i, \delta) \in I \times \kappa_n$, let $b_{i,\delta}$ be the partial map sending $X \in D$ to the unique $\zeta < \kappa_n$ such that $(\pi[X], \zeta)\ \pi(R_i)\ (a^*, \delta)$ if such ζ exists. By definition of system, every $b_{i,\delta}$ is an R_i-branch. Condition (2) of Definition 4.3 is satisfied as well: indeed, if $X \in D$, then by condition (3) of Definition 4.1, there exists $\zeta, \eta < \kappa_n$ and $i \in I$ such that $(\pi[X], \zeta)\ \pi(R_i)\ (a^*, \eta)$, hence $X \in \mathrm{dom}(b_{i,\eta})$. It remains to prove that for some $j \in J$, b_j is cofinal. For every $X \in D$, we fix i_X, δ_X such that $X \in \mathrm{dom}(b_{i_X,\delta_X})$. The set I has size less than κ_m in W, moreover ν^+ is regular and $\mathbb{C}_{tail}$ does not add sequences of length less than κ_m, so there are $D' \subseteq D$ cofinal and i, δ in $V[H^*]$ such that $i = i_X$ and $\delta = \delta_X$, for every $X \in D$. This means that $X \in \mathrm{dom}(b_{i,\delta})$ for every $X \in D$, namely $b_{i,\delta}$ is a cofinal branch. □

By the Preservation Theorem, a cofinal R_i-branch b exists in W, for some $i \in I$. Assume that $i = (a, b, \mu)$, for every $X \subseteq Y$ in $\mathrm{dom}(b)$, we have $(a, b) \Vdash (X, b(X)) <_{\dot{F}_\mu} (Y, b(Y))$. If $G_0 \times G_1 \subseteq \mathbb{L}(\mu)$ is any generic filter containing the condition (a, b), then the branch b determines a cofinal branch for $\dot{F}_\mu$ in $W[G_0 \times G_1]$, contradicting the fact that $\dot{F}_\mu$ is a name for a (ν^+, λ)-tree with no cofinal branches. □

References

[1] J. Cummings and M. Foreman. The Tree Property, *Advances in Mathematics* 133: 1-32 (1998).

[2] L. Fontanella. Strong Tree Properties for Two Successive Cardinals, to appear in the *Archive for Mathematical Logic*.

[3] T. Jech. Some Combinatorial Problems Concerning Uncountable Cardinals, *Annals of Mathematical Logic* 5: 165-198 (1972/73).

[4] A. Kanamori. *The Higher Infinite. Large Cardinals in Set Theory from their Beginnings*, Perspectives in Mathematical Logic. Springer-Verlag, Berlin, (1994).

[5] R. Laver. Making the supercompactness of κ indestructible under κ-directed closed forcing, *Israel Journal of Mathematics* 29: 385-388 (1978).

[6] M. Magidor. Combinatorial Characterization of Supercompact Cardinals, *Proc. American Mathematical Society* 42: 279-285 (1974).

[7] W. J. Mitchell, Aronszajn Trees and the Independence of the Transfer Property, *Annals of Mathematical Logic* 5: 21-46 (1972).
[8] M. Magidor and S. Shelah. The Tree Property at Successors of Singular Cardinals, *Archive for Mathematical Logic* 35 (5-6): 385-404 (1996).
[9] I. Neeman. The Tree Property up to $\aleph_{\omega+1}$ (submitted).
[10] D. Sinapova. The Tree Property at $\aleph_{\omega+1}$, to appear in the Journal of Symbolic Logic (2012).
[11] M. Viale. On the Notion of Guessing Model, to appear in the *Annals of Pure and Applied Logic*.
[12] M. Viale and C. Weiss. On the Consistency Strength of the Proper Forcing Axiom, submitted.
[13] C. Weiss. Subtle and Ineffable Tree Properties, Phd thesis, *Ludwig Maximilians Universitat Munchen* (2010)
[14] C. Weiss. The Combinatorial Essence of Supercompactness, submitted to the *Annals of Pure and Applied Logic*.

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